# Notes on Category Theory

Rodrigo Ramos[1]

**About these notes**

This is a personal notebook that I use to keep my notes on category theory. I mostly use Coecke (2008); Coecke and Paquette (2009); Awodey (2010) as main references for the basic contents. These notes' purpose is to understand how quantum mechanics is formulated in a category-theoretical language (Heunen and Vicary, 2019), and, in particular, how the Choi-Jamiołkowski isomorphism looks like in this framework.

São Paulo
2nd April 2024

[1] rirm@ifi.unicamp.br

# Contents

Chapter One

# *The Basics*

**Chapter Contents**



## 1 Motivation

From my perspective, category theory is a fascinating topic in and of itself: it allows us to make statements that hold true in different mathematical fields without the need to commit to the specific structures of one of them. More remarkable than that is the fact that we can formulate quantum theory in a category-theoretical language (Heunen and Vicary, 2019). Since quantum theory is a particular kind of a generalized probabilistic theory (GPT), that means we can also describe GPTs within this framework and doing so would help me get a better understanding of how one could formulate a Choi-Jamiołkowski-esque isomorphism in GPTs without resorting to purification (Chiribella et al., 2010). With that in mind and without further ado, let us embark on our category studies.

## 2 Categories

**Definition 1.1: Category**

A category $\mathbf{C}$ consists of

1. A family of objects: $A, B, C, \ldots$
2. For any two objects $A$ and $B$, a set of arrows (also called morphisms) $\mathbf{C}(A, B)$. For each arrow $f \in \mathbf{C}(A, B)$, we write it as $A \xrightarrow{f} B$.
3. For any objects $A$,$B$ and $C$, a composition rule
$$\circ : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C),\ (f, g) \mapsto g \circ f,\ such\ that$$
4. For any $f \in \mathbf{C}(A, B)$,$g \in \mathbf{C}(B, C)$ and $h \in \mathbf{C}(C, D)$ the composition is associative:
$$h \circ (g \circ f) = (h \circ g) \circ f;$$
5. For any object $A$ corresponds an arrow $1_A \in \mathbf{C}(A, A)$, called the identity arrow and it satisfies
$$f = f \circ 1_A = 1_B \circ f$$

for any $f \in \mathbf{C}(A, B)$.

By a long shot, that is the most important definition in this entire text. In order to get a better picture of what a category is, a few examples might help.

**Example 1.1: Sets**

Take the class of all sets as our family of objects, consider the functions between sets as our arrows and let $\circ$ be the ordinary composition rule for functions. For each set $A$, define $A \xrightarrow{1_A} A$ to be the identity map on $A$, *i.e.* $1_A \doteq \mathrm{id}_A$. These data constitute the category of sets, which is written as **Sets**.

**Example 1.2: Concrete categories**

- The category $\mathbf{FdVect}_{\mathbb{K}}$ consists of
  1. Finite-dimensional vector spaces over $\mathbb{K}$ as objects;
  2. Linear maps between vector spaces as arrows;
  3. Ordinary composition of linear maps as $\circ$;
  4. For each object $V$, the identity map $\mathrm{id}_V$ as $1_V$.
- The category **Pos** consists of
  1. Partially ordered sets as objects;
  2. Monotone maps as arrows, *i.e.* $a \leqslant a' \implies f(a) \leqslant f(a')$;
  3. Ordinary composition of functions and identity maps as $\circ$ and $1_A$, respectively.
- The category **Rel** consists of
  1. Sets $A, B, C, \ldots$ as objects;
  2. Relations $R \subset A \times B$ as morphisms;
  3. A composition rule that maps $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ into the relation
  $$\{(a, c) \in A \times C \mid \exists b \in B : (a, b) \in R, (b, c) \in S\};$$
  4. Identity morphisms $1_A = \{(a, a) \in A \times A \mid a \in A\}$ for every $A \in \mathcal{O}bj(\mathbf{Rel})$.

A more physically inclined example of category is

**Example 1.3: Physical Processes**

The category **PhysProc** consists of

1. All physical systems $A, B, C, \ldots$ as objects;
2. All physical processes which take a physical system $A$ into another physical system $B$ as morphisms $A \longrightarrow B$, and
3. Sequential composition of physical processes as $\circ$, and the process that leaves system $A$ invariant as $1_A$.

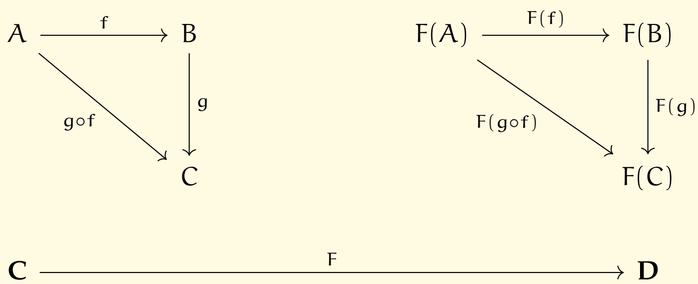
If we want to construct a category whose objects are categories themselves, then we need to tell what are the arrows $\mathbf{C} \xrightarrow{F} \mathbf{D}$ that take a category $\mathbf{C}$ to another category $\mathbf{D}$. For our purposes, the notion of *functor* originates from this necessity.

**Definition 1.2: Functor**

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and denote the family of their objects by $\mathcal{O}bj(\mathbf{C})$ and $\mathcal{O}bj(\mathbf{D})$, respectively. A functor $F: \mathbf{C} \to \mathbf{D}$ consists of

1. A mapping
$$F : \mathcal{O}bj(\mathbf{C}) \to \mathcal{O}bj(\mathbf{D}),\ A \mapsto F(A) \text{ and}$$
2. For any $A, B, C \in \mathcal{O}bj(\mathbf{C})$, a mapping
$$F: \mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B)),\ f \mapsto F(f)$$
that preserves identites and compositions.

In terms of diagrams, functors behave as follows:




Functor composition works as expected, that is, if $F : \mathbf{C} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{E}$ are functors, then we define the functor $G \circ F: \mathbf{C} \to \mathbf{E}$ via the maps that define $F$ and $G$, *i.e*

$$A \in \mathcal{O}bj(\mathbf{C}) \mapsto (G \circ F)(A) \doteq G(F(A)) \in \mathcal{O}bj(\mathbf{E}), \text{ and}$$

$$f \in \mathbf{C}(A, B) \mapsto (G \circ F)(f) \doteq G(F(f)) \in \mathbf{E}(G(F(A)), G(F(B))).$$

It is fairly easy to show that the functor composition as defined above is associative, and preserves morphisms compositions and identities. Furthermore, for each category $\mathbf{C}$, one can construct an identity functor $1_{\mathbf{C}}$. From all that, we have that the collection of all categories and all functors constitutes a category which we denote by **Cat**[1].

### Groups

Due to Noether's theorem (be its classical version or its quantum-mechanical one), we know that groups play a fundamental role in physics. Hence, it seems like a very basic request to have a category-theoretical description of groups if we want to talk about symmetries in quantum mechanical systems in a category-theoretical fashion. In order to do this, we first have to introduce the notion of a *monoid*.

**Definition 1.3: Monoid**

A monoid is a set $M$ equipped with a binary relation $\cdot : M \times M \to M$ that is associative and admits a unit, *i.e.* for all $x, y, z \in M$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

[1] I did not mention the composition rule nor the identity, but they should be fairly obvious at this point.

and there is an (unique) element $1 \in M$ such that

$$1 \cdot x = x \cdot 1 = x.$$

A simple example of a monoid is the set of arrows from $A$ to $A$, denoted by $\mathrm{Hom}_{\mathbf{C}}(A, A)$, where $A$ is an object of a category **C**. Besides, given that monoid homomorphisms preserve the monoid structure, we can construct the category **Mon** whose objects are monoids and whose arrows are monoid homomorphisms. What is more interesting is that we can interpret a monoid as a category itself, as it is illustrated in the following example.

**Example 1.4: Monoids as categories**

If $M$ is a monoid, we can identify it with a category **M** that has a single object $*$, whose morphisms consists are arrows $* \xrightarrow{m} *$, where $m \in M$. The composition between arrows is given by the monoid product $\cdot$ and the identity $* \xrightarrow{1_*} *$ is associated to the unit element $1 \in M$.

Now, recall that a group $G$ is just a monoid (in the sense of definition 1.3) such that every $g \in G$ admits an (unique) inverse $g^{-1} \in G$. So, at the category theory level, we would expect groups to the monoids (in the sense of example 1.4) whose arrows, in a sense, also have inverses. This is made precisely clear through the following definition.

**Definition 1.4: Isomorphism**

Let **C** be a category. Two objects $A, B \in \mathcal{Obj}(\mathbf{C})$ are isomorphic if there are morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that

$$g \circ f = 1_A \text{ and } f \circ g = 1_B.$$

In this case, $f$ is called an isomorphism and $g \equiv f^{-1}$ is called the inverse of $f$.

**Example 1.5: Groups as categories**

Putting together example 1.4 and definition 1.4, we conclude that a group $G$ is a category with one object and whose morphisms are all isomorphisms.

## 3 Building new categories

Now that it is clear what a category is, we can construct new ones. Let us progressively walk from very simple and intuitive examples towards somewhat more elaborate ones.

**Example 1.6: Product Category**

Given two categories **C** and **D**, the product category $\mathbf{C} \times \mathbf{D}$ has consists of

1. Objects of the form $(A, B)$ where $A \in \mathcal{Obj}(\mathbf{C})$ and $B \in \mathcal{Obj}(\mathbf{D})$
2. Morphisms of the form

$$(A, B) \xrightarrow{(f,g)} (A', B')$$

where $f \in \mathbf{C}(A, A')$ and $g \in \mathbf{D}(B, B')$

3. A composition rule $\circ$ defined componentwise using the composition rules from **C** and **D**, *i.e*,
$$(f', g') \circ (f, g) \doteq (f' \circ f, g' \circ g).$$
4. A unit morphism $1_{(A,B)} \doteq (1_A, 1_B)$ for each object $(A, B)$.

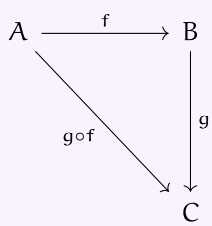
I like to think that the product category is a generalisation of the Cartesian product between sets.

**Exercise 1.1.** *Construct two projection functors in* $\mathbf{C} \times \mathbf{D}$.

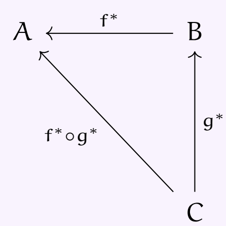
**Example 1.7: Dual Category**

Given a category **C**, we construct its dual (or opposite) category, denoted by $\mathbf{C}^{op}$, by setting

1. Its objects as the same ones from **C**,
2. Its morphisms are morphisms in **C**, but with interchanged domain and codomain. Illustratively, a morphism
$$B \xleftarrow{f^*} A$$
in $\mathbf{C}^{op}$ is a morphism
$$A \xrightarrow{f} B$$
in **C**. Objects in $\mathbf{C}^{op}$ are written with a superscript "*" so that they are not mistaken by objects in the original category. In the first diagram above, we would write $f^* \in \mathbf{C}^{op}(A^*, B^*)$.
3. The composition rule is $f^* \circ g^* \doteq (g \circ f)^*$. It can be represented by flipping the arrows in the original category, a diagram




in **C**, means



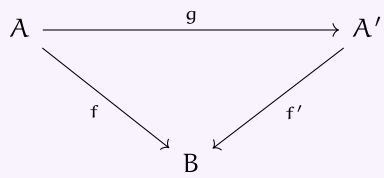

in $\mathbf{C}^{op}$.

4. To each object $A^*$, the identity is $1_{A^*} \doteq (1_A)^*$.

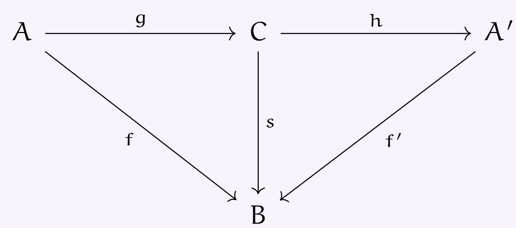
**Example 1.8: Slice Category**

Let **C** be a category and $A \in \mathcal{O}bj(\mathbf{C})$. The slice category of **C** over B consists of

1. All morphisms f such that $\text{cod}(f) = B$ as objects,
2. Given two objects $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B$, a morphism between then is a morphism $A \xrightarrow{g} A'$ such that the following diagram commutes:



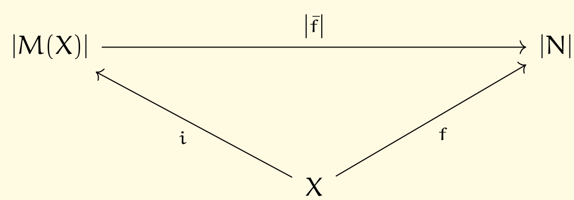

3. Given morphisms $g$ from $A \xrightarrow{f} B$ to $C \xrightarrow{s} B$ and $h$ from $C \xrightarrow{s} B$ to $A' \xrightarrow{f'} B$, their composition is the morphism $h \circ g$ such that the following diagram commutes:




4. To each object $A \xrightarrow{f} B$, the associated identity morphism is $A \xrightarrow{1_A} A$ because it makes the appropriate diagrams commute (AFAIU).

## 4 Universal Properties

In this section, we present three elementary examples of what is known as "universal property". As I understand it, an universal property tells us how to single out a specific object[2] within a category by stating what are the unique properties that define such object. In the case of free monoids, as we will see, given a set $X$, the free monoid over $X$ is the monoid $M(X)$ such that for any monoid $N$ and any function $f : X \to |N|$, there exists an unique monoid homomorphism $\bar{f} : M(X) \to N$ such that the following diagram commutes:



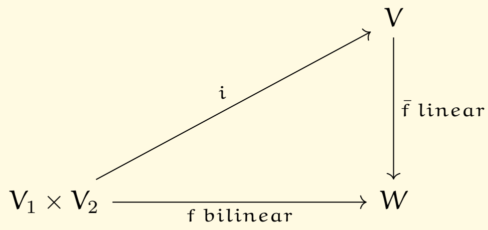

where $|M(X)|$ and $|N|$ are the underlying sets of the respective monoids and $i : X \to |M(X)|$ is the inclusion map.

### Tensor Product of Vector Spaces

This first example is of utmost importance to everyone who wishes to study quantum theory. Let $V_1, V_2$ and $W$ be real vector spaces and consider a bilinear map $f : V_1 \times V_2 \to W$. The core idea behind what is a tensor product is to find a real vector space $V$ such that

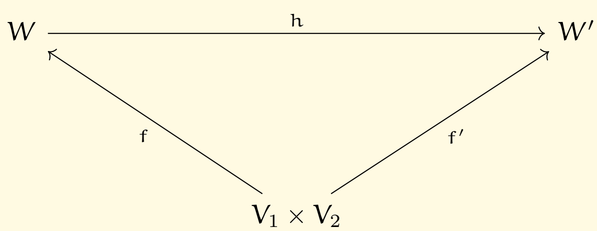
1. We can insert $V_1 \times V_2$ into $V$ by means of an inclusion map $i : V_1 \times V_2 \to V$ and
2. We can define an unique linear map $\bar{f} : V \to W$ such that the following diagram commutes:

[2] Up to isomorphism!!

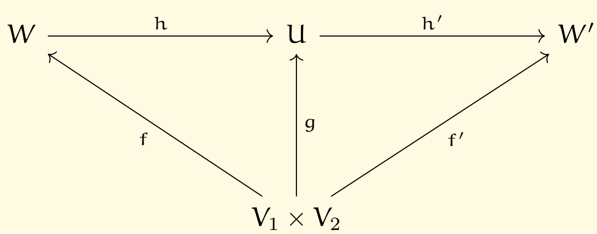



Let us express this idea in a more categorical fashion. First, fix the vector spaces $V_1$ and $V_2$ (we are interested in constructing their tensor product). We define the category **Bl(**$V_1 \times V_2$**)** by saying that

1. Its objects are bilinear maps $V_1 \times V_2 \to *$ (this symbolises their codomain is any real vector space),
2. For any two objects $f : V_1 \times V_2 \to W$ and $f' : V_1 \times V_2 \to W'$, which we will denote by $(f, W)$ and $(f', W')$, respectively, a morphism $(f, W) \xrightarrow{h} (f', W')$ is a linear map $h : W \to W'$ such that the following diagram commutes:




3. Given morphisms $(f, W) \xrightarrow{h} (g, U)$ and $(g, U) \xrightarrow{h'} (f', W')$, their composition is the morphism $(f, W) \xrightarrow{h' \circ h} (f', W')$ such that the following diagram commutes




4. The identity morphism associated to $(f, W)$ is the identity map $\mathrm{id}_W : W \to W$.

Now, the conditions 1) and 2) above tell us what are the desirable characteristics that the tensor product $V$ should possess; however, we do not want them to hold only for some particular $W$. That is why we needed to define $\mathbf{Bl}(V_1 \times V_2)$. Hence, in categorical terms we have the following definition:

**Definition 1.5: Universal Property of Tensor Product of Vector Spaces**

Let $V_1$ and $V_2$ be real vectors spaces. The tensor product of $V_1$ and $V_2$ is the object $(i, V_1 \otimes V_2) \in \mathcal{O}bj(\mathbf{Bl}(V_1 \times V_2))$ such that for any object $(f, W) \in \mathcal{O}bj(\mathbf{Bl}(V_1 \times V_2))$, there exists a unique morphism $\bar{f} : (i, V_1 \otimes V_2) \to (f, W)$.

In these terms, the tensor product of vectors spaces is some object in $\mathcal{O}bj(\mathbf{Bl}(V_1 \times V_2))$ from which arrows depart from. More formally, we say that $(i, V_1 \otimes V_2)$ is an **initial object**.

**Definition 1.6: Initial Object**

Let **C** be a category. We say an object $0 \in \mathcal{O}bj(\mathbf{C})$ is initial if for any object $C \in \mathcal{O}bj(\mathbf{C})$ there is a unique morphism
$$0 \longrightarrow C.$$

For the sake of completeness, we also have the notion of **terminal objects**.

**Definition 1.7: Terminal Object**

Let **C** be a category. We say an object $1 \in \mathcal{O}bj(\mathbf{C})$ is terminal if for any object $C \in \mathcal{O}bj(\mathbf{C})$ there is a unique morphism
$$C \longrightarrow 1.$$

Now, does such an object in fact exists? We only gave a definition that characterizes the tensor product, but it could be that there is no such $(i, V_1 \otimes V_2)$. It would be somewhat lengthy to construct such $V_1 \otimes V_2$, but I am sure one can find some math book, YouTube lecture, lecture notes or even a Math Stack Exchange post showing how it is done. I myself tried to do this here.

**Exercise 1.2.** *Prove that if such $(i, V_1 \otimes V_2)$ exists, then it is unique up to isomorphism.*

**Exercise 1.3.** *More generally, prove that initial and terminal objects are unique up to isomorphism.*

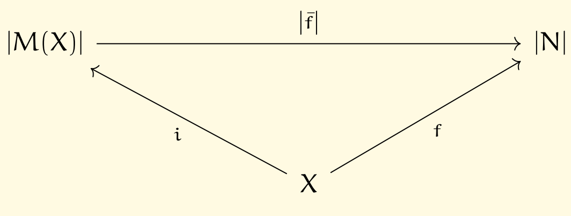
### Free Monoids

If $X$ is a set, a finite sequence of elements of $X$ is a map $f : N \to X$, where $N \subset \mathbb{N}$ is finite. We denote a finite sequence $f$ on $X$ by a string $x_1x_2\dots x_n$, where $n \in \mathbb{N}$ is the cardinality of $N$. Let

$$|M(X)| \doteq \{x_1x_2\dots x_n : n \in \mathbb{N}, x_i \in X, \ i = 1, 2\dots, n\}$$

be the set of all such finite sequences on $X$. We can turn $|M(X)|$ into a monoid (which we denote by $M(X)$) by setting that the composition of two finite sequences $x_1\dots x_n$ and $y_1\dots y_m$ is the sequence $z_1\dots z_n z_{n+1} z_{n+m}$, where $z_i = x_i$ for $1 \leqslant i \leqslant n$ and $z_{n+i} = y_i$ for $1 \leqslant i \leqslant m$. Clearly, there is an inclusion map $i : X \to |M(X)|$, which takes $x \in X$ to its obvious finite sequence.

**Exercise 1.4.** *Show that the monoid $M(X)$ as constructed above satisfies the following universal property: for any monoid $N$ and any function $f : X \to |N|$, there exists an unique monoid homomorphism $\bar{f} : M(X) \to N$ such that the following diagram commutes:*




*where $|M(X)|$ and $|N|$ are the underlying sets of the respective monoids and $i : X \to |M(X)|$ is the inclusion map.*

### Free Categories

## 5 Products

## 6 Duality

## 7 Natural Transformations

Chapter Two

# *Monoidal Categories*

**Chapter Contents**

# Bibliography

Bob Coecke. Introducing categories to the practicing physicist, 2008.

Bob Coecke and Eric Oliver Paquette. Categories for the practising physicist, 2009.

Steve Awodey. *Category theory*, volume 52. Oxford University Press, 2010.

Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford University Press, Oxford, November 2019. doi: 10.1093/oso/9780198739623.001.0001.

Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Probabilistic theories with purification. *Physical Review A*, 81(6), June 2010. ISSN 1094-1622. doi: 10.1103/physreva.81.062348. URL http://dx.doi.org/10.1103/PhysRevA.81.062348.